On Circuit Diameter Bounds via Circuit Imbalances

Zhuan Khye (Cedric) Koh

Joint work with Daniel Dadush, Bento Natura, László A. Végh

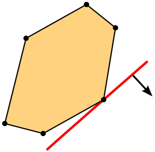
Overview

1. Diameter and Circuit Diameter of Polyhedra
2. Circuit Imbalance Measure κ
3. Main Results:
   - Circuit diameter bounds in terms of κ
   - Circuit augmentation algorithms for LP (runtime in terms of κ)
4. Conclusion and Future Directions

Linear Programming

- Given $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$, $c \in \mathbb{R}^n$,

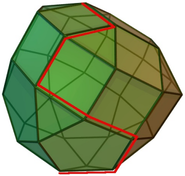
$$\text{(Primal)} \quad \min\ c^\top x \quad \text{s.t. } Ax = b,\ x \geq 0 \qquad \text{(Dual)} \quad \max\ b^\top y \quad \text{s.t. } A^\top y \leq c$$



- n variables, m constraints, $L_{A,b,c}$ encoding length of the input
- Ellipsoid and Interior Point Methods run in $poly(n, m, L_{A,b,c})$ time.
- For Simplex Method, no polynomial pivot rule is known.

Smale's 9th Question: Is there a strongly polynomial algorithm for LP? (runs in $poly(n, m)$ arithmetic operations using $poly(n, m, L_{A,b,c})$ space.)

Simplex Method and Diameter



- Consider the polyhedron

$$P = \{x : Ax = b, x \geq 0\}.$$

- The diameter of P is the max # of edges in a shortest path between any 2 vertices.

- Diameter $\leq$ # of simplex pivots.

Hirsch's Conjecture: The diameter of a d-dimensional polyhedron with f facets is $\leq f - d$.

- Disproved by Santos in 2012.
- Polynomial version ($\leq poly(f)$) is still open.

- Quasi-polynomial upper bound known [Kalai–Kleitman '92], [Todd '14]. Current best bound: $(f-d)^{\log O(d/\log d)}$ [Sukegawa '18].

Circuit Diameter

- For $W = \ker(A)$, $g \in W \setminus \{0\}$ is a circuit if $\nexists\, h \in W \setminus \{0\}$ such that

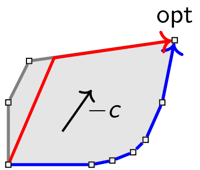
$$\mathrm{supp}(h) \subsetneq \mathrm{supp}(g).$$

- Circuits are all possible edge directions of P when varying b.

- A circuit walk is a sequence of points $x^{(1)}, x^{(2)}, \ldots, x^{(k)}$ in P where $x^{(t+1)}$ is obtained from $x^{(t)}$ by moving along a circuit maximally.

- The circuit diameter of P is the max # of steps in a shortest circuit walk between any 2 vertices.

- Circuit diameter $\leq$ diameter.



Circuit Hirsch's Conjecture [Borgwardt, Finhold, Hemmecke '15]**:** The circuit diameter of a d-dimensional polyhedron with f facets is $\leq f - d$.

Goals

1. Upper bound the circuit diameter of P.

2. Give an efficient circuit augmentation algorithm for solving

$$\min\{c^\top x : x \in P\}.$$

State of the Art

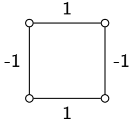
- Circuit diameter of combinatorial polyhedra:
  - Dual transportation polyhedra [Borgwardt, Finhold, Hemmecke '15].
  - Matching, travelling salesman and fractional stable set polytopes [Kafer, Pashkovich, Sanità '19].
- Circuit augmentation algorithms in network flow:
  - Edmonds–Karp–Dinic algorithm for max flow.
  - Min-mean cycle canceling for min-cost flow [Goldberg–Tarjan '89].
  - Min-ratio cycle canceling for min-cost flow [Wallacher '89].
  - All extended to LP [Bland '76], [Gauthier–Desrosiers '21], [Ekbatani, Natura, Végh '21], [McCormick–Shioura '00].
- Other circuit augmentation rules [De Loera, Hemmecke, Lee '15].
- Circuit augmentation on 0/1-polytopes [De Loera, Kafer, Sanità '19].

Circuit Imbalance Measure

- For $W = \ker(A)$, the circuit imbalance measure is

$$\kappa_W = \max\left\{\left|\frac{g_i}{g_j}\right| : i,j \in \operatorname{supp}(g), g \text{ circuit in } W\right\}.$$

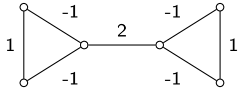
- $\kappa_W = \kappa_{W^\perp}$.

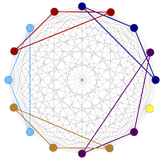


- If A is totally unimodular, then $\kappa_W = 1$.

- For integral A with max subdeterminant Δ, $\kappa_W \leq \Delta$.
  - E.g., if A is the incidence matrix of K_n, then $\kappa_W \leq 2$ and $2^{\lfloor n/3 \rfloor} \leq \Delta$ because

$$\det\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix} = 2.$$





Circuit Diameter Bounds

Theorem 1 (Uncapacitated)

For $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$, the circuit diameter of

$$P = \{x : Ax = b, x \geq 0\}$$

is $O(m^2 \log(m + \kappa_W))$.

Theorem 2 (Capacitated)

For $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$ and $u \in \mathbb{R}^n$, the circuit diameter of

$$P_u = \{x : Ax = b, 0 \leq x \leq u\}$$

is $O(m^2 \log(m + \kappa_W) + n \log n)$.

Note: Simple reduction from P_u to P only gives $O(n^2 \log(n + \kappa_W))$.

Uncapacitated Circuit Diameter

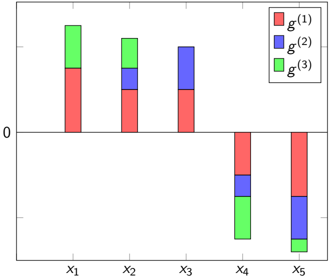
- Let $P = \{x : Ax = b, x \geq 0\}$ and $W = \ker(A)$.
- Every $x \in W$ can be decomposed into $\leq n$ conformal circuits $g^{(i)} \in W$:

$$x = \sum_{i=1}^{k} g^{(i)}$$

such that $\text{sgn}(g_j^{(i)}) = \text{sgn}(x_j)$ for all $i \in [k]$ and $j \in [n]$.

Idea: Given vertices $x, y \in P$, let B be a basis for y and $N := [n] \setminus B$.

- Decompose $y - x$ into conformal circuits $g^{(1)}, g^{(2)}, \ldots, g^{(k)}$.
- Pick $g^{(i)}$ with the largest $\|g_N^{(i)}\|_1$ and augment.
- Repeat.



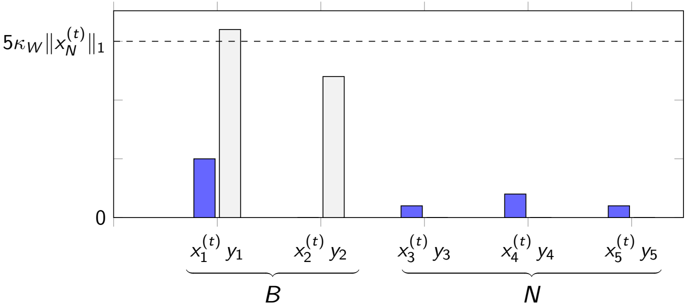
Proof of the $O(m^2 \log(m + \kappa_W))$ bound

- Let $x^{(0)}, x^{(1)}, \ldots$ be the resulting circuit walk from x to y.

Geometric decay: $\|x_N^{(t+1)}\|_1 \leq \left(1 - \frac{1}{n}\right) \|x_N^{(t)}\|_1$

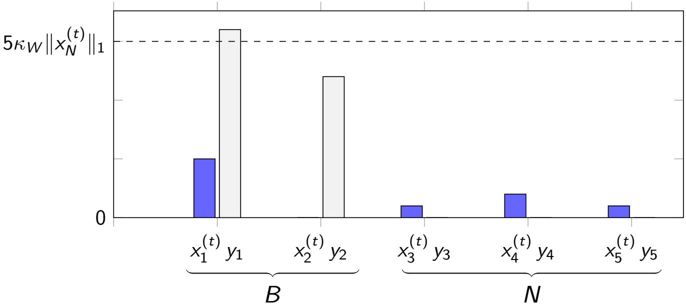
- We analyze the sets

$$L_t = \left\{ i \in [n] : y_i > n\kappa_W \|x_N^{(t)}\|_1 \right\} \qquad R_t = \left\{ i \in [n] : x_i^{(t)} \leq n y_i \right\}$$



Proof of the $O(m^2 \log(m + \kappa_W))$ bound

$$L_t = \left\{ i \in [n] : y_i > n\kappa_W \|x_N^{(t)}\|_1 \right\} \qquad R_t = \left\{ i \in [n] : x_i^{(t)} \leq n y_i \right\}$$



Lemma 1: $L_t \subseteq L_{t+1} \subseteq B$ and $R_t \subseteq R_{t+1}$.

Lemma 2: L_t or R_t grows in $O(n \log(n + \kappa_W))$ iterations.

$\implies$ Circuit diameter $O(n^2 \log(n + \kappa_W))$. How to turn n into m?

Capacitated Circuit Diameter

- Let $P_u = \{x : Ax = b, 0 \leq x \leq u\}$.

- First, augment using conformal circuits of $y - x^{(t)}$ which minimize an auxiliary cost function.

- Then, remove $n - m$ capacity constraints using the following subroutine:

Support-Circuit (a.k.a. Moving to a face)

Input: Feasible point $x \in P_u$

Output: A circuit $z \in \ker(A)$ where $\text{supp}(z) \subseteq \text{supp}(x)$.

- Finally, reduce to uncapacitated form and apply Theorem 1.

$$\left\{(x, s) : \begin{bmatrix} A & 0 \\ I_m & I_m \end{bmatrix} \begin{bmatrix} x \\ s \end{bmatrix} = \begin{bmatrix} b \\ u \end{bmatrix}, (x, s) \geq 0 \right\}$$

Circuit Augmentation Algorithms

- Recall $P = \{x : Ax = b, x \geq 0\}$. We use the following subroutine:

Ratio-Circuit (Wallacher's Rule)

Input: Cost $c \in \mathbb{R}^n$ and feasible point $x \in P$.

Output: A circuit $z \in \ker(A)$ such that $z_{[n]\setminus \mathrm{supp}(x)} \geq 0$ and minimizes

$$\frac{c^\top z}{\sum_{i \in \mathrm{supp}(x)} \frac{\max(-z_i, 0)}{x_i}}.$$

Geometric decay: Augmenting along z shrinks the optimality gap by a factor of $(1 - \frac{1}{n})$.

$\Longrightarrow$ Weakly polynomial but not finite!

Circuit Augmentation Algorithms

Theorem 3 (Feasibility)

There is a circuit augmentation algorithm which given $N \subseteq [n]$ finds

$$x \in P \text{ such that } x_N = 0,$$

or a dual certificate showing that no such solution exists, using $O(n^2 \log(n + \kappa_W))$ Ratio-Circuit and n^2 Support-Circuit calls.

Theorem 4 (Optimization)

There is a circuit augmentation algorithm which solves

$$\min\{c^\top x : x \in P\}$$

using $O(n^3 \log(n + \kappa_W))$ Ratio-Circuit and n^3 Support-Circuit calls.

- How does this differ to the circuit diameter setting?

Conclusion

For $\min\{c^\top x : Ax = b, x \geq 0\}$ where $A \in \mathbb{R}^{m \times n}$,

Summary of Results:

1. Circuit diameter bounds:
   - $O(m^2 \log(\kappa_W + m))$ for uncapacitated.
   - $O(m^2 \log(\kappa_W + m) + n \log n)$ for capacitated.

2. Circuit augmentation algorithms:
   - $O(n^2 \log(\kappa_W + n))$ Ratio-Circuit and n^2 Support-Circuit calls for feasibility.
   - $O(n^3 \log(\kappa_W + n))$ Ratio-Circuit and n^3 Support-Circuit calls for optimization.

Open Problems:

1. $poly(n, m, \log \kappa_W)$ bound on the diameter? $poly(n, m, L_{A,b})$?
   - Current best: $O((n-m)^3 m \kappa_W \log(\kappa_W + n))$ [Dadush–Hähnle '16].
2. $poly(n, m)$ bound on the circuit diameter?