An Efficient Characterization of Submodular Spanning Tree Games

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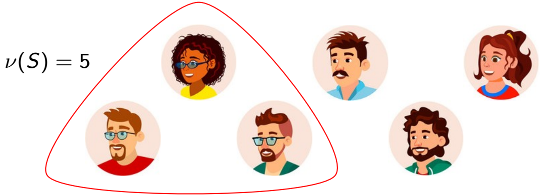

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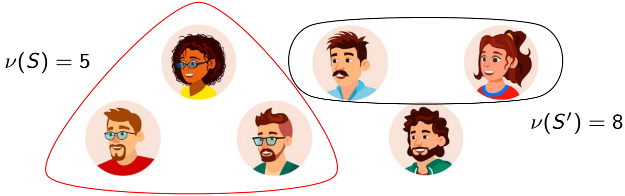

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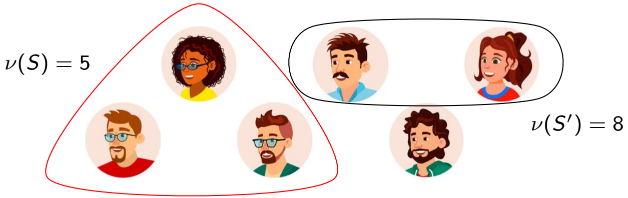

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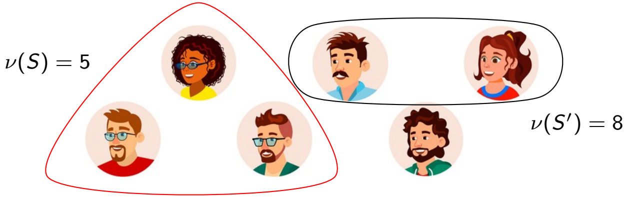

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- An outcome of the game is an allocation $y \in \mathbb{R}^N$ such that

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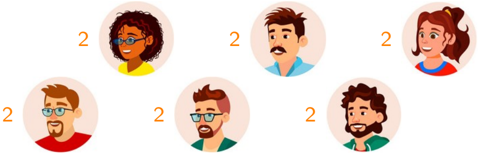

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- Generally hard to compute unless ν satisfies certain properties.

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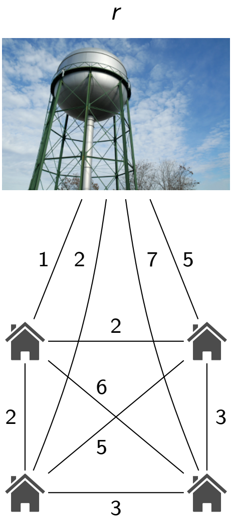

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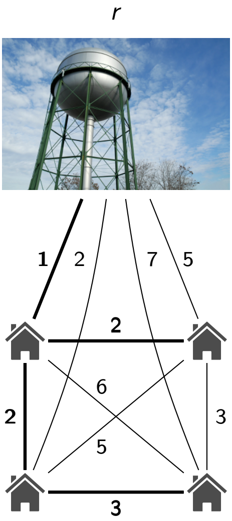

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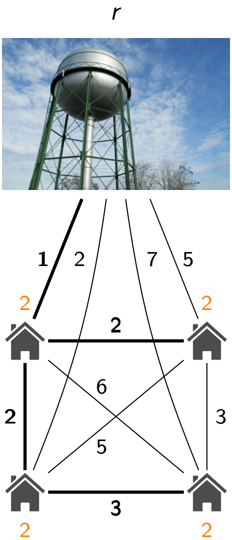

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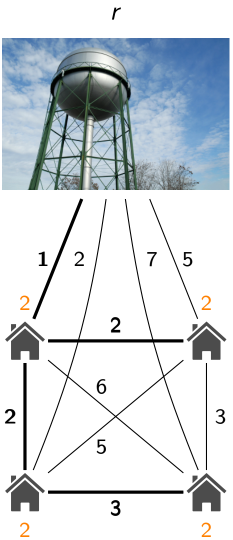

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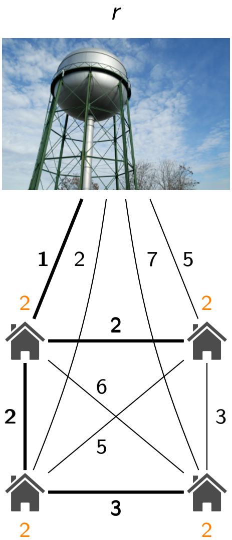

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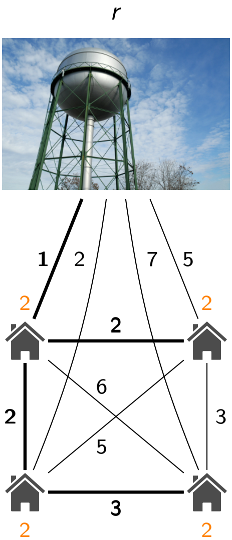

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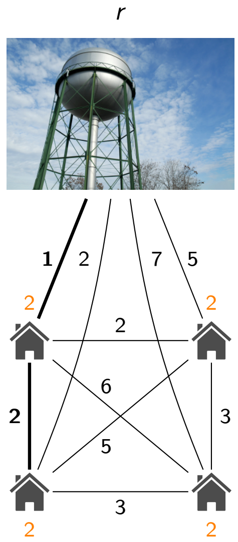

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Can we find an **efficient** *characterization of submodular instances?*

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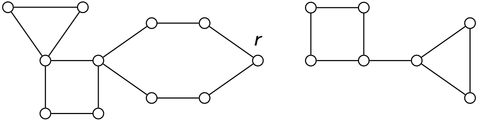

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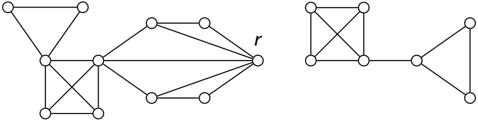

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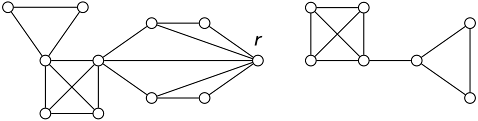

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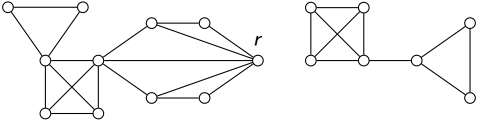

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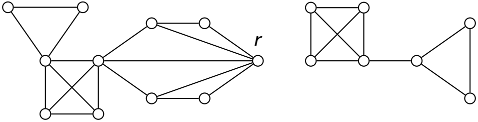

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• In this work, we **fully** characterize submodular instances. This characterization can be verified in **polynomial time**.

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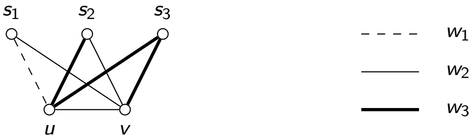

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Main result

Theorem: The spanning tree game on G is submodular if and only if:

1. There are no violated cycles in G_i for all $i < k$.
2. For every candidate edge uv, $f_{uv}(\hat{N}(uv)) \geq 0$.

Furthermore, these conditions can be verified in polynomial time.

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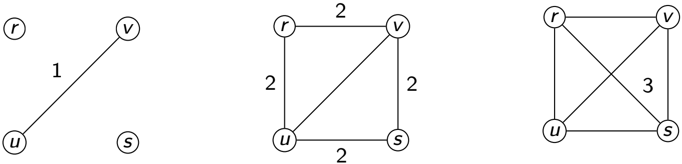

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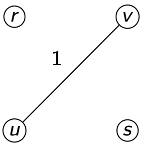

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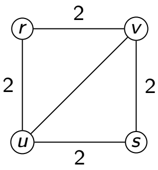

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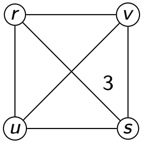

$G_3 = G$

- G_2 violates the condition, yet the instance is submodular.

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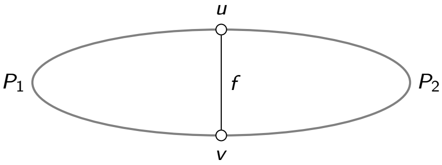

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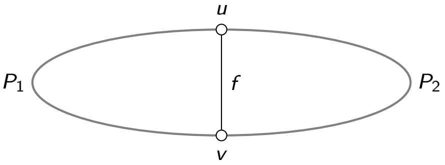

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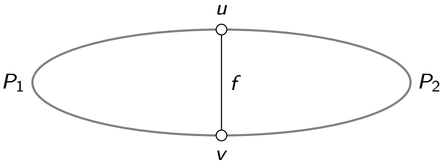

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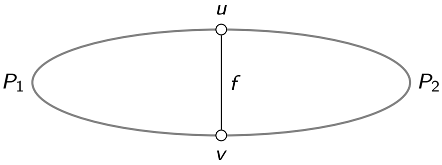

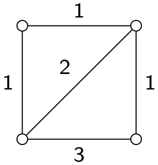

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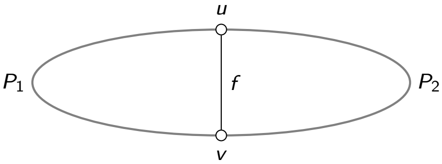

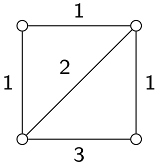

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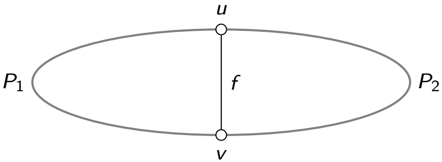

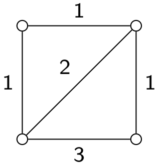

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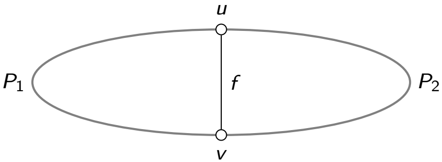

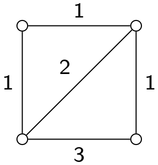

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- Coincides with [Kobayashi & Okamoto '14] when $k = 2$.

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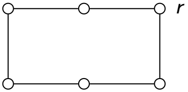

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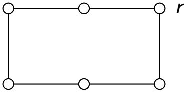

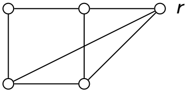

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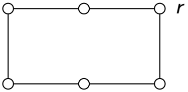

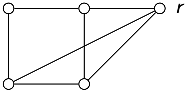

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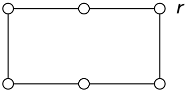

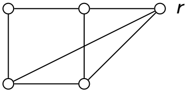

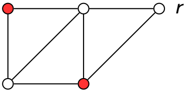

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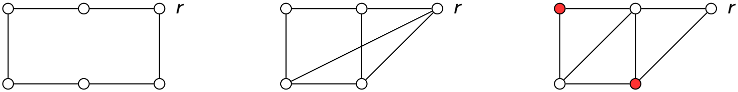

Lemma 2: If the instance is submodular, then there are no bad holes or bad induced diamonds in G_i for all $i < k$.

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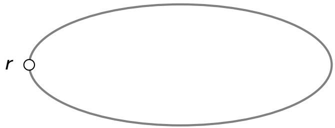

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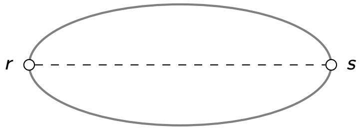

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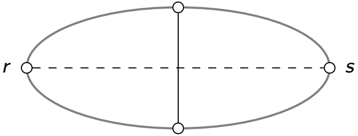

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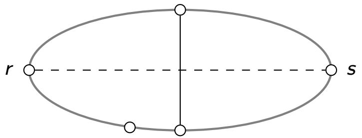

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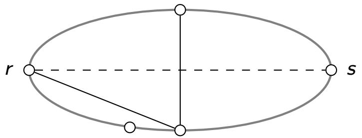

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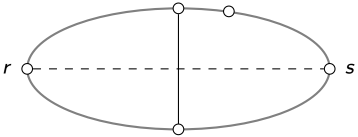

Violated cycles

Lemma 1: If the instance is submodular, then there are no violated cycles in G_i for all $i < k$.

Proof: Contrapositive.

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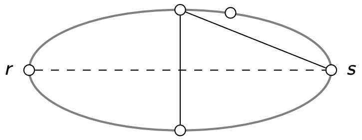

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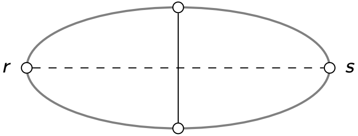

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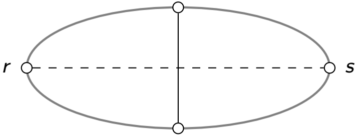

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Remark: Violated cycles can be detected in polynomial time.

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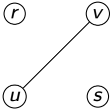
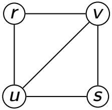
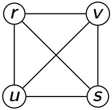


G_1



G_2



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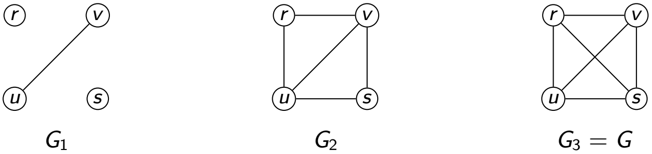

$$f_{uv}(\{r,s\}) = \mathrm{mst}(\{r,s,u\}) + \mathrm{mst}(\{r,s,v\}) - \mathrm{mst}(\{r,s\}) - \mathrm{mst}(\{r,s,u,v\})$$

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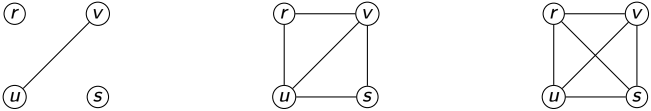

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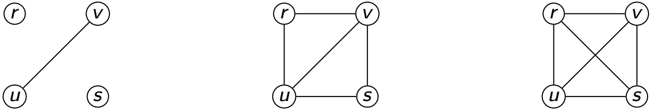

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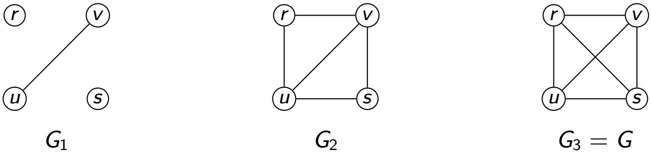

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Lemma 5*: Assume $f_{uv}(\hat{N}(uv)) \geq 0$ for every candidate edge uv. Then, f_{uv} is inclusion-wise nonincreasing in $\hat{N}(uv)$ for every candidate edge uv.

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Theorem: The spanning tree game on G is submodular if and only if:

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Thank you!